RADIAL SOLUTIONS OF LANE-EMDEN-FOWLER EQUATIONS WITH PUCCI'S EXTREMAL OPERATORS
SOLUZIONI RADIALI DI EQUAZIONI TIPO LANE-EMDEN-FOWLER PER GLI OPERATORI ESTREMALI DI PUCCI

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Abstract. We report on some recent results obtained for positive radial solutions of Lane-Emden-Fowler type equations with Pucci's operators as principal parts. The presented results include the asymptotic analysis of almost critical solutions in the unit ball, existence results in annular domains and sharp Liouville-type results for exterior Dirichlet problems.

Sunto. Vengono presentati alcuni risultati recenti riguardanti soluzioni radiali positive di equazioni tipo Lane-Emden-Fowler aventi gli operatori di Pucci come parte principale. I risultati includono un'analisi asintotica per soluzioni quasi-critiche nella palla unitaria, teoremi di esistenza per domini anulari e teoremi di tipo Liouville ottimali per problemi di Dirichlet in domini esterni.

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1. Introduction

We report on some recent results from [2, 11, 12] about positive radial solutions of the Lane-Emden-Fowler type equations

$$-\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) = u^p \tag{1}$$

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where $p > 1$ and $\mathcal{M}^{\pm}_{\lambda,\Lambda}$ are the Pucci extremal operators. For fixed constants $\Lambda \geq \lambda > 0$ we set

$$\mathcal{M}^{-}_{\lambda,\Lambda}(X) = \inf_{\lambda I \leq A \leq \Lambda I} \operatorname{tr}(AX) = \lambda \sum_{\mu_i > 0} \mu_i + \Lambda \sum_{\mu_i < 0} \mu_i$$

$$\mathcal{M}^{+}_{\lambda,\Lambda}(X) = \sup_{\lambda I \leq A \leq \Lambda I} \operatorname{tr}(AX) = \Lambda \sum_{\mu_i > 0} \mu_i + \lambda \sum_{\mu_i < 0} \mu_i$$

$\mu_1, \ldots, \mu_n$ being the eigenvalues of any squared symmetric matrix X.

Pucci's extremal operators, acting as barriers in the whole class of operators with fixed ellipticity constants $\lambda \leq \Lambda$, are the prototype of fully nonlinear uniformly elliptic operators and they play a crucial role in the regularity theory for fully nonlinear elliptic equations, see [4].

We recall that if $\Lambda = \lambda$, both Pucci's operators reduce, up to a multiplicative factor, to the Laplace operator. Thus, the semilinear equation $-\Delta u = u^p$ is included as a very special case of the problems we are considering.

Associated with the operators $\mathcal{M}^{\pm}_{\lambda,\Lambda}$, there are the dimension like parameters

$$\tilde{n}_- := \tfrac{\Lambda}{\lambda}(n-1) + 1 \quad \text{for } \mathcal{M}^{-}_{\lambda,\Lambda}\,,$$

$$\tilde{n}_+ := \tfrac{\lambda}{\Lambda}(n-1) + 1 \quad \text{for } \mathcal{M}^{+}_{\lambda,\Lambda}\,,$$

which are referred to as "intrinsic" or "effective" dimensions of the considered operators. Let us observe that one has always $\tilde{n}_+ \leq n \leq \tilde{n}_-$, and equalities hold true if and only if $\Lambda = \lambda$.

In some previously studied cases, the numbers $\tilde{n}_\pm$ are known to determine the optimal threshold for the exponent p separating the existence from the nonexistence range for solutions of equation (1). Namely, in [5] the case of entire supersolutions was considered, by proving that, in the case $\tilde{n}_\pm > 2$,

$$\exists\, u > 0\,,\ -\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2 u) \geq u^p \quad \text{in } \mathbb{R}^n \Longleftrightarrow p > \frac{\tilde{n}_\pm}{\tilde{n}_\pm - 2}\,,$$

and, in the case $\tilde{n}_\pm \leq 2$, no positive supersolutions exist for any $p > 0$. In the sequel, we will always assume that $\tilde{n}_\pm > 2$. The same threshold has been proved in [1] to be optimal for the existence of solutions in any exterior domain, that is

$$\exists\, u>0\,,\ -\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u)=u^p \quad \text{in } \mathbb{R}^n\setminus K \Longleftrightarrow p>\frac{\tilde{n}_\pm}{\tilde{n}_\pm-2}\,,$$

where $K\subset\mathbb{R}^n$ is any nonempty compact set. In the above results, supersolutions are meant in the viscosity sense and any symmetry property on u is required.

The above Liouville type results may be used to obtain a-priori estimates and consequent existence results for positive solutions of Dirichlet problems posed in bounded domains, see [15]. However, these are obtained under the restriction $p\leq\frac{\tilde{n}_\pm}{\tilde{n}_\pm-2}$ which is not the optimal threshold on p for the existence of solutions. For existence results in half-spaces we refer to [14].

Better existence results under optimal thresholds on the exponent p can be obtained if we restrict our attention to radially symmetric solutions. In the radial setting, the existence of entire positive solutions has been studied in [9], where it has been proved that there exist two critical exponents p^*_+ and p^*_- associated with $\mathcal{M}^+_{\lambda,\Lambda}$ and $\mathcal{M}^-_{\lambda,\Lambda}$ respectively, such that

$$\exists\, u \text{ radial}, u>0\,,\ -\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u)=u^p \quad \text{in } \mathbb{R}^n \Longleftrightarrow p\geq p^*_\pm\,.$$

The dependence of the radial critical exponents on the effective dimensions is not explicitly known, but they are shown to satisfy, when $\lambda<\Lambda$,

$$\frac{\tilde{n}_-+2}{\tilde{n}_--2}<p^*_-<\frac{n+2}{n-2}<p^*_+<\frac{\tilde{n}_++2}{\tilde{n}_+-2}\,.$$

Thus, the critical exponents $p^*_\pm$ play in the fully nonlinear radial setting the same role played by the well known Sobolev exponent $\frac{n+2}{n-2}$ in the semilinear case, with p^*_+ (p^*_-) being subcritical (respectively, supercritical) with respect to the intrinsic dimensions.

The analysis of [9] yields, as a byproduct, the dual result on the existence of positive solutions of Dirichlet problems in balls, namely

$$\exists\, u>0\,,\ -\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u)=u^p \quad \text{in } B,\ u=0 \ \text{ on } \partial B \Longleftrightarrow p<p^*_\pm\,.$$

Note that, in this case, the radial symmetry of the solutions is not a restriction, since, by [6], any positive solution in the ball is radial. The critical exponents $p^*_\pm$, therefore, give the optimal thresholds for the existence of positive solutions in balls and, as proved in [8], also in domains sufficiently closed to balls.

We are concerned here with some further results recently obtained for radial solutions. In particular, we report in Section 2 on the asymptotic analysis performed in [2] about almost critical solutions in balls, that are solutions of the Dirichlet problems

$$
\begin{cases} -\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u^{\pm}_{\epsilon}) = (u^{\pm}_{\epsilon})^{p^*_{\pm}-\epsilon} \ \text{ in } B\,, \\ \\ u^{\pm}_{\epsilon} > 0 \ \text{ in } B\,,\ u^{\pm}_{\epsilon} = 0 \ \text{ on } \partial B\,, \end{cases} \tag{2}
$$

In the semilinear case $\Lambda = \lambda$, the problem has been largely studied, and it is well known that the unique positive radial solutions u_ϵ blow up and concentrate at the center of the ball as $\epsilon \to 0$, while the energy satisfies

$$
J(u_\epsilon) = \left(\frac{1}{2} - \frac{1}{p_\epsilon + 1}\right) \int_B u_\epsilon^{p_\epsilon+1}\,dx \ \stackrel{\epsilon\to 0}{\longrightarrow} \ \frac{1}{n} S^n,
$$

where

$$
S = \inf_{u \in H^1(\mathbb{R}^n)\setminus\{0\}} \frac{\|Du\|_{L^2}}{\|u\|_{L^{2^\star}}}
$$

is the best constant in the Sobolev inequality. The local profile of u_ϵ, suitably rescaled, is that of

$$
U(x) = \frac{1}{\left(1 + \frac{|x|^2}{n(n-2)}\right)^{\frac{n-2}{2}}}
$$

the Talenti's function which realizes the constant S, see [16], and satisfies the limiting equation

$$
-\Delta U = U^{p_*} \quad \text{ in } \mathbb{R}^n
$$

We obtained in [2] an extension of the above results to the fully nonlinear framework. Solutions $u^{\pm}_{\epsilon}$ of problems (2) are proved to concentrate at the center of B, blowing up at the center and vanishing locally uniformly outside the center, while a suitably defined related energy is preserved, despite the lack of a variational structure in the problem.

In Section 3 we present the results obtained in [11] about the existence, for any $p > 1$, of positive radial solutions of Dirichlet problems in annular domains, as well as the more recent sharp Liouville type results for exterior Dirichlet problems obtained in [12]. Once again the results are obtained through a careful analysis of the ODE associated with radial solutions, and despite the fact that the critical radial exponents $p^*_{\pm}$ are not explicitly known.

As a general final remark, let us observe that, when considering radial solutions of problems (1), because of the dependence of operators $\mathcal{M}^{\pm}_{\lambda,\Lambda}$ on the sign of the eigenvalues of the hessian matrix, one obtains an ODE with discontinuous coefficients, having jumps at points where the solution changes its monotonicity and/or its concavity. This is a feature of the fully nonlinear problem which makes the previous techniques developed in the semilinear case not directly applicable, and requires sometimes new ad hoc proofs.

2. Radial solutions for Pucci's operators

In this section we recall some well known facts about radial solutions of Lane-Emden-Fowler equations relative to Pucci's operators. So, let us focus on positive radial solutions $u(x) = u(|x|)$ of the elliptic equation (1).

Without loss of generality, we consider C^2 classical solutions, since, even if we look at weak solutions in the viscosity sense, the elliptic regularity theory immediately applies. We then observe that the eigenvalues of the hessian matrix $D^2u(x)$ of a smooth radial function u are nothing but $u''(r)$, which is simple, and $\frac{u'(r)}{r}$, which has multiplicity $n-1$, where $r = |x|$.

Hence, taking into account the expression of Pucci's operators in dependence of the sign of the eigenvalues, we can easily obtain the ODE satisfied by any radial solution according to its monotonicity and convexity properties.

In particular, recalling the expression of the intrinsic dimensions $\tilde{n}_{\pm}$, one obtains that u is a radial solution of $-\mathcal{M}^{+}_{\lambda,\Lambda}(D^2u) = u^p$ if and only if u satisfies

$\mathbf{C_1}$: if $u'(r) \geq 0$ and $u''(r) \leq 0$, then

$$u''(r) + (\tilde{n}_- - 1)\frac{u'(r)}{r} = -\frac{u^p(r)}{\lambda}\,;$$

$\mathbf{C_2}$: if $u'(r) \leq 0$ and $u''(r) \leq 0$, then

$$u''(r) + (n-1)\frac{u'(r)}{r} = -\frac{u^p(r)}{\lambda}\,;$$

$\mathbf{C_3}$: if $u'(r) \leq 0$ and $u''(r) \geq 0$, then

$$u''(r) + (\tilde{n}_+ - 1)\frac{u'(r)}{r} = -\frac{u^p(r)}{\Lambda}\,.$$

Let us emphasize that any solution u cannot be at the same time positive, convex and increasing in any interval, since otherwise $-\mathcal{M}^+_{\lambda,\Lambda}(D^2u) < 0 < u^p$. In particular, any critical point of u is a local strict maximum point for u.

For radial solutions of the equation $-\mathcal{M}^-_{\lambda,\Lambda}(D^2u) = u^p$ one obtains analogous ODEs as above, with λ and Λ interchanged.

In any case, the differential problem satisfied by a radial solution u can be written as a second oder ODE of the form

$$u''(r) = \mathcal{G}\left(\frac{u'(r)}{r}, -u^p(r)\right) \tag{3}$$

with $\mathcal{G}$ Lipschitz continuous in $\mathbb{R}^2$. Conversely, solutions of suitable Cauchy problem associated with the ODE (3) will produce radially symmetric solutions for Pucci's operators. We recall that the existence and uniqueness of maximal solutions of Cauchy problems for equation (3) can be obtained by the classical Peano-Picard's Theorem and by approximation arguments needed to handle the singularity at $r = 0$, see [7, 13].

As in the semilinear case, a powerful tool for studying the problem is the so called Emden-Fowler transformation (see [10]), which reduces the initial problem to an autonomous equation. Setting

$$x(t) = r^{\frac{2}{p-1}} u(r)\,, \quad r = e^t\,, \tag{4}$$

a direct computation shows that

$$\begin{aligned} u' \geq 0 \quad &\Longleftrightarrow \quad x' \geq \frac{2}{p-1} x\,, \\ u'' \leq 0 \quad &\Longleftrightarrow \quad x' \geq \frac{2}{p-1} x - \frac{x^p}{\lambda\,(n-1)}\,. \end{aligned}$$

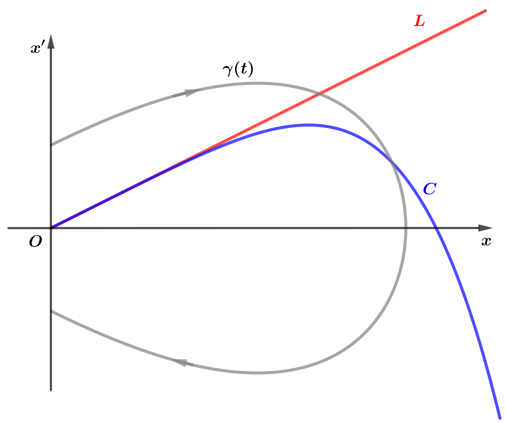
Hence, the new unknown x satisfies

$$\begin{cases} x'' = \tilde{a}_- x' + 2\tilde{b}_- x - \dfrac{x^p}{\lambda} & \text{if } \ x' \geq \dfrac{2}{p-1} x \\ x'' = a x' + 2b x - \dfrac{x^p}{\lambda} & \text{if } \ \dfrac{2}{p-1} x \geq x' \geq \dfrac{2}{p-1} x - \dfrac{x^p}{\lambda\,(n-1)} \\ x'' = \tilde{a}_+ x' + 2\tilde{b}_+ x - \dfrac{x^p}{\Lambda} & \text{if } \ x' \leq \dfrac{2}{p-1} x - \dfrac{x^p}{\lambda\,(n-1)} \end{cases} \tag{5}$$

with coefficients defined respectively as

$$
\begin{aligned}
a &= \frac{n+2-(n-2)p}{p-1}\,, &\qquad \tilde a_\pm &= \frac{\tilde n_\pm +2-(\tilde n_\pm -2)p}{p-1}\,,\\
b &= \frac{2\left((n-2)p-n\right)}{(p-1)^2}\,, &\qquad \tilde b_\pm &= \frac{2\left((\tilde n_\pm -2)p-\tilde n_\pm\right)}{(p-1)^2}\,.
\end{aligned}
\tag{6}
$$

Associated with a solution x of the above problem, one can consider the trajectory $\gamma(t) = (x(t), x'(t))$ in the phase plane. Thus, the coefficients of the autonomous equation satisfied by x are piecewise constant and jump whenever the trajectory γ intersects either the half-line $L = \left\{x' = \frac{2}{p-1}x\right\}$ or the curve $\mathcal{C} = \left\{x' = \frac{2}{p-1}x - \frac{x^p}{\lambda\,(n-1)}\right\}$, see the picture below.



3. Solutions in balls or in the whole space

Looking for radial solutions of equation (1) in balls or in the whole space is equivalent to looking at solutions of the ODE (3) for $r > 0$, equipped with the initial conditions

$$u(0) = \alpha > 0\,,\ \ u'(0) = 0$$

for any $\alpha > 0$. The resulting Cauchy problem has a unique solution u_α defined on a maximal interval $[0, \rho(\alpha)]$, with $\rho(\alpha) \le +\infty$ and $u_\alpha(\rho_\alpha) = 0$ if $\rho(\alpha) < +\infty$. By a scaling invariance of the equation, it is easy to see that

$$u_\alpha(r) = \alpha\, u_1(\alpha^{\frac{p-1}{2}} r)\,,$$

so that the fact that $\rho(\alpha)$ is finite or not, only depends on p.

To the Emden-Fowler tranformed $x_\alpha(t)$ of u_α, according to (4), is associated in the phase plane a trajectory $\gamma_\alpha(t)$ exiting from the origin for $t=-\infty$ and either intersecting the $x'-$axis in a finite time or staying in the right half-plane for all $t\in\mathbb{R}$. In this latter case, by the Poncaré-Bendixon theorem, for $t\to+\infty$, $\gamma_\alpha(t)$ will either approximate a periodic orbit or converge to one of the two equilibrium points $(0,0)$ and $\left((2\Lambda\tilde b_+)^{\frac{1}{p-1}},0\right)$, see (5) and (6).

This problem has been carefully analyzed by Felmer and Quaas [9], who classified the solutions u_α according to the asymptotic behavior of $\gamma_\alpha(t)$ for $t\to+\infty$. Following the terminology used in [9], for an entire solutions $u_\alpha(r)$ defined in $[0,+\infty)$ we say that

- u_α is a *fast decaying* solution if $\gamma_\alpha(t)\to(0,0)$, that is if
$$\lim_{r\to+\infty} r^{\frac{2}{p-1}}u_\alpha(r)=0\,;$$
- u_α is a *slow decaying* solution if $\gamma_\alpha(t)\to\left((2\Lambda\tilde b_+)^{\frac{1}{p-1}},0\right)$, that is if
$$\lim_{r\to+\infty} r^{\frac{2}{p-1}}u_\alpha(r)=(2\Lambda\tilde b_+)^{\frac{1}{p-1}}\,;$$
- u_α is a *pseudo-slow decaying* solution if $\gamma_\alpha(t)$ approaches a periodic orbit, that is if
$$0<\liminf_{r\to+\infty} r^{\frac{2}{p-1}}u_\alpha(r)<\limsup_{r\to+\infty} r^{\frac{2}{p-1}}u_\alpha(r)<+\infty\,.$$

The deep analysis of [9] yields the existence of unique critical exponents $p_*^\pm$, for $\mathcal{M}^+_{\lambda,\Lambda}$ and $\mathcal{M}^-_{\lambda,\Lambda}$ respectively, such that entire radial solutions of equation (1) do exist if and only if $p\geq p_*^\pm$. The critical exponents $p_*^\pm$ are characterized as the unique exponents for which there exist fast decaying solutions $(u_\alpha)^*_\pm$, whereas for $p>p_*^\pm$ solutions may be slow or pseudo-slow depending on p. Moreover, for $p=p_*^\pm$, the stable-unstable manifold theorem applied to the trajectories $\gamma_\alpha(t)$ yields the vanishing order of $x_\alpha(t)$ and $x'_\alpha(t)$. In terms of any critical solution $(u_\alpha)^*_\pm$, this implies that there exists a positive constant $c_\alpha>0$ such that
$$\lim_{r\to\infty} r^{\tilde n_\pm-2}(u_\alpha)^*_\pm(r)=c_\alpha\,,\ \lim_{r\to\infty} r^{\tilde n_\pm-1}((u_\alpha)^*_\pm)'(r)=-(\tilde n_\pm-2)c_\alpha\,.$$
More than that, precise pointwise estimates on solutions $(u_\alpha)^*_\pm$ are given in [2]. We state the results for $\alpha=1$, and we refer to [2] for the proof.

Theorem 3.1. *Let $u_\pm^*$ be the unique positive radial solution of*

$$-\mathcal{M}^\pm_{\lambda,\Lambda}(D^2u_\pm^*) = (u_\pm^*)^{p_*^\pm} \quad \text{in } \mathbb{R}^n\,,\ u_\pm^*(0) = 1\,.$$

There exist unique $r_0^\pm > 0$ such that $(u_\pm^*)''(r_0^\pm) = 0$ and positive constants $c^\pm < C^\pm$ such that, for any $r \geq r_0^\pm$, one has

$$\frac{u_\pm^*(r_0^\pm)}{\left(1 + C^\pm(r^2 - (r_0^\pm)^2)\right)^{\frac{\tilde n_\pm - 2}{2}}} \leq u_\pm^*(r) \leq \frac{u_\pm^*(r_0^\pm)}{\left(1 + c^\pm(r^2 - (r_0^\pm)^2)\right)^{\frac{\tilde n_\pm - 2}{2}}}\,. \tag{7}$$

Let us emphasize that, for r sufficiently large, $u_\pm^*$ is a solution of

$$(u_\pm^*)''(r) + (\tilde n_\pm - 1)\frac{(u_\pm^*)'(r)}{r} = -\frac{(u_\pm^*)^{p_\pm^*}(r)}{\Lambda}\,,$$

while the functions appearing on the left and right hand side of (7) satisfy

$$v''(r) + (\tilde n - 1)\frac{v'(r)}{r} = -Av^{\frac{\tilde n + 2}{\tilde n - 2}}\,,$$

for a suitable constant $A > 0$. Thus, any form of comparison principle may be applied in order to prove (7), which is obtained instead by using monotonicity properties of a Pohozaev-type energy functional associated with the solution $u_\pm^*$.

As a dual result, the analysis of [9] about radial entire solutions produces a sharp existence result for the Dirichlet problems

$$\begin{cases} -\mathcal{M}^\pm_{\lambda,\Lambda}(D^2u) = u^p \ \text{ in } B\,, \\ \\ u > 0 \ \text{ in } B\,,\ u = 0 \ \text{ on } \partial B\,, \end{cases} \tag{8}$$

where B is, say, the unit ball in $\mathbb{R}^n$ centered at the origin. We recall that, for any exponent $p > 1$, positive solutions of problem (8), when exist, are radially symmetric and radially decreasing, since the moving plane technique of Gidas, Ni and Nirenberg applies (see [6]). For other partial symmetry results in the fully nonlinear framework (also for sign changing solutions), see [3].

Therefore, one obtains that problem (8) admits solution if and only if $p < p_\pm^*$.

In the recent paper [2], we analyzed the asymptotic behaviour as $\epsilon \to 0$ of the almost critical solutions $u_\epsilon^\pm$ of problems (2), by proving that a concentration phenomenon occurs. Precisely, we have the following result, for the proof of which we refer to [2]. In the statement below, we drop the subscripts $\pm$ in order to simplify the notations, and we set $p_\epsilon = p^*_\pm - \epsilon$.

Theorem 3.2. *As $\epsilon \to 0$, the solutions u_ϵ satisfy:*

- $M_\epsilon := u_\epsilon(0) \to +\infty$;
- $u_\epsilon(x) \to 0$ *in* $C^2_{\mathrm{loc}}(B \setminus \{0\})$;
- $M_\epsilon^{\frac{p_\epsilon(\tilde n-2)-\tilde n}{2}} u_\epsilon(x) \to c_0 \left(\frac{1}{|x|^{\tilde n-2}} - 1\right)$ *for all* $x \in B \setminus \{0\}$;
- $\int_B u_\epsilon^{p_\epsilon+1}(x) g_{u_\epsilon,p_\epsilon}(x)\, dx \;\to\; \int_{\mathbb{R}^n} (u^*)^{p^*+1}(x) g_{u^*,p^*}(x)\, dx$, *for suitable radial weights* $g_{u_\epsilon,p_\epsilon}$ *and* g_{u^*,p^*} *associated with the solutions* u_ϵ *and* u^* *respectively.*

We observe that, taking advantage of the scaling invariance of the equation, one can define the rescaled functions

$$\tilde u_\epsilon(x) = \frac{1}{M_\epsilon} u_\epsilon \left(\frac{x}{M_\epsilon^{\frac{p_\epsilon-1}{2}}} \right)$$

which satisfy

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2\tilde u_\epsilon) = \tilde u_\epsilon^{p_\epsilon} \text{ in } B_{M_\epsilon^{\frac{p_\epsilon-1}{2}}}\,, \\ \\ \tilde u_\epsilon = 0 \text{ on } \partial B_{M_\epsilon^{\frac{p_\epsilon-1}{2}}}\,. \end{cases}$$

The functions $\tilde u_\epsilon$ are uniformly bounded, being $\|\tilde u_\epsilon\|_\infty = \tilde u_\epsilon(0) = 1$. By elliptic estimates, there is a sequence $\epsilon_k \to 0$ such that

$$\tilde u_{\epsilon_k} \to u^* \quad \text{ in } C^2_{\mathrm{loc}}(\mathbb{R}^n)$$

where u^* is the entire fast decaying solution satisfying $u^*(0) = 1$. Scaling back $\tilde u_\epsilon$, we have

$$u_\epsilon(x) = M_\epsilon \tilde u_\epsilon(M_\epsilon^{p_\epsilon - 1} r)\,,$$

so that the asymptotic analysis of u_ϵ is equivalent to the asymptotic analysis at infinity of $\tilde u_\epsilon$. This is performed through a careful analysis of the ODE satisfied by $u_\epsilon, \tilde u_\epsilon$ and u^*, and by relying on the estimates given by Theorem 3.1, despite the fact that the limiting exponent p^* and function u^* are not explicitly known.

4. Solutions in annuli or in exterior domains

In this section we consider positive radial solutions, vanishing on the boundary, for equations (1) posed in possibly unbounded annular domains. Fixing the inner radius at, say, 1, this amounts to consider solutions $u_\alpha(r)$ of the ODE (3) for $r > 1$, satisfying the initial conditions

$$u_\alpha(1) = 0\,, \quad u'_\alpha(1) = \alpha > 0\,. \tag{9}$$

The solution u_α is positive on a maximal interval $[1, \rho(\alpha))$, with $1 < \rho(\alpha) \leq +\infty$, and $u_\alpha(\rho(\alpha)) = 0$ if $\rho(\alpha) < +\infty$. We observe that, differently from the case considered in the previous section, the current problem is no more invariant by rescaling, so that $\rho(\alpha)$ can be finite or not in dependence of α and p.

In the recent paper [11], again through a careful analysis of the ODE (3) and some maximum principle arguments, we proved that, for any $p > 1$, if α is sufficiently large, then $\rho(\alpha) < +\infty$. This yields the following existence result, whose proof is given in [11].

Theorem 4.1. *Let $A_{a,b} = \{a < |x| < b\}$ be any bounded annular domain in $\mathbb{R}^n$. Then, for any $p > 1$, there exist positive solutions of the Dirichlet problems*

$$\begin{cases} -\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) = u^p \quad \text{in } A_{a,b}\,, \\ \\ u > 0 \quad \text{in } A_{a,b}\,,\; u = 0 \quad \text{on } \partial A_{a,b}\,, \end{cases} \tag{10}$$

The existence result of Theorem 4.1 is actually proved in [11] for general uniformly elliptic and invariant by rotations operators. Moreover, it also gives existence of negative solutions and, by gluing together the positive and the negative ones, one can obtain sign changing solutions, see [11].

In the specific case of Pucci's operator, we investigated further on the solutions u_α satisfying the initial conditions (9) in the recent work [12]. This further analysis shows that the critical exponents $p^*_\pm$ play a role also in the existence of solutions for exterior Dirichlet problems. Indeed, we have the following Liouville-type result for exterior domains.

Theorem 4.2. *There exist positive radial solutions of problems*

$$-\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) = u^p \;\text{ in } \mathbb{R}^n \setminus \overline{B}\,, \quad u = 0 \;\text{ on } \partial B$$

if and only if $p > p^*_\pm$.

Moreover, for any fixed $p > p^*_\pm$, *one has*

- *there exist infinitely many solutions;*
- *there exists a unique fast decaying solution.*

A particularly delicate step in the proof of Theorem 4.2 is the proof of the nonexistence of solutions in the critical cases $p = p^*_\pm$, as well as of the uniqueness of the fast decaying solutions, which is obtained in [12] by using different arguments for $\mathcal{M}^+_{\lambda,\Lambda}$ and $\mathcal{M}^-_{\lambda,\Lambda}$. For $\mathcal{M}^-_{\lambda,\Lambda}$, the fact that $p^*_- > \frac{\tilde{n}_-+2}{\tilde{n}_--2}$ is exploited, jointly with some properties of solutions of supercritical semilinear problems. For $\mathcal{M}^+_{\lambda,\Lambda}$, whose critical exponent satisfies $p^*_+ < \frac{\tilde{n}_++2}{\tilde{n}_+-2}$, a different proof is obtained as an application of Gauss-Green Theorem in the phase-plane.

Let us finally remark that the existence of positive solutions for exterior Dirichlet problems is naturally related to the existence of sign changing radial solutions in balls or in the whole space (see also Remark 5.2 in [11]). We plan to study the existence of such solutions and their asymptotic behavior in a forthcoming work.

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